

A SIMPLE COMPARISON BETWEEN SKOROKHOD & RUSSO-VALLOIS INTEGRATION FOR INSIDER TRADING

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ABSTRACT. We consider a simplified version of the problem of insider trading in a financial market. We approach it by means of anticipating stochastic calculus and compare the use of the Skorokhod and the Russo-Vallois forward integrals within this context. We conclude that, while the forward integral yields results with a clear financial meaning, the Skorokhod integral does not provide a suitable formulation for this problem.

1. INTRODUCTION

The stochastic differential equation

$$(1) \quad \frac{dx}{dt} = f(x) + g(x) \xi(t),$$

where $\xi(t)$ is a “white noise”, is a mathematical model with applications in many disciplines [9]. The precise meaning of this equation is found via the introduction of a suitable stochastic integral, that can be either Itô:

$$(2) \quad dx = f(x) dt + g(x) dB_t,$$

where B_t is a Brownian motion, Stratonovich:

$$(3) \quad dx = f(x) dt + g(x) \circ dB_t,$$

or yet another option [10]. The mathematical theory for stochastic differential equations of Itô or Stratonovich type has been constructed [15] and both problems are shown to be well-posed under reasonable conditions, then minimizing from a pure mathematics viewpoint the difference between them. However, from an applied viewpoint the difference between equations (2) and (3) can be dramatic, as both may lead to radically different dynamics [9]. Which *interpretation of noise* is chosen depends on modeling, that is, on the particular application which mathematical treatment leads to equation (1). Perhaps because of this, a vast literature on which is the right interpretation does exist [12].

One of the main applications of the theory of stochastic differential equations is the study of financial markets. Lets consider a classical financial market with one asset free of risk (the bond)

$$(4) \quad \begin{aligned} dS_0 &= \rho S_0 dt, \\ S_0(0) &= M_0, \end{aligned}$$

and a risky asset (the stock) modeled by Geometric Brownian motion

$$(5) \quad \begin{aligned} dS_1 &= \mu S_1 dt + \sigma S_1 dB_t, \\ S_1(0) &= M_1, \end{aligned}$$

where the constants $M_0, M_1, \rho, \mu, \sigma \in \mathbb{R}^+ :=]0, \infty[$ have the following financial meaning:

- M_0 is the initial wealth to be invested in the bond.
- M_1 is the initial wealth to be invested in the stock.

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- ρ is the interest rate of the bond.
- μ is the appreciation rate of the stock.
- σ is the volatility of the stock.

The total initial wealth is $M = M_0 + M_1$ and we assume that $\mu > \rho$. We consider the trader possesses a fixed total initial wealth M at the initial time $t = 0$ and is free to choose what fraction of it, M_0 and M_1 , is invested in each asset. Clearly, at any time $t > 0$, the total wealth is given by

$$S(t) = S_0(t) + S_1(t).$$

We will consider this financial market on $[0, T]$ for a fixed future time $T > 0$. Then we have the following result.

Theorem 1.1. *The expected value of the total wealth at time T is*

$$\mathbb{E}[S(T)] = M_0 e^{\rho T} + M_1 e^{\mu T}.$$

Proof. Using Itô calculus we solve equations (4) and (5) to find

$$\begin{aligned} S_0(t) &= M_0 e^{\rho t}, \\ S_1(t) &= M_1 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\}, \end{aligned}$$

and then the expectation of $S(t)$ at time $t = T$ is

$$\begin{aligned} \mathbb{E}[S(T)] &= \mathbb{E}[S_0(T)] + \mathbb{E}[S_1(T)] \\ &= M_0 \mathbb{E}[e^{\rho T}] + M_1 \mathbb{E} \left[\exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\ &= M_0 e^{\rho T} + M_1 e^{\mu T}. \end{aligned}$$

□

Any trader that wants to maximize the expected wealth at time T should obviously choose the strategy

$$M_0 = 0, \quad M_1 = M,$$

what in turn yields the maximal expected wealth

$$\mathbb{E}[S(T)] = M e^{\mu T}.$$

Some remarks are now in order. First of all, this maximization problem may be regarded as a toy model for the Merton portfolio optimization problem [13]. Indeed, everything here becomes simplified due to the absence of a utility function modeling risk aversion. This function has not been introduced for two reasons: to keep our approach and results as simple as possible, and also for some modeling reasons that will be specified in the next section. Additionally, it is important to remark that problem (5) represents an easy example of the resolution of the *Itô versus Stratonovich dilemma* referred to in the first paragraph of this Introduction. In our modeling of the stock price evolution we assumed that μ is the expected rate of return of the risky asset. Therefore, this assumption together with the martingale property of the Itô integral, they impose unambiguously that (5) is an Itô stochastic differential equation. Things will be different in the next section, in which the trader will be assumed to possess at time $t = 0$ additional information with respect to the one contained in the filtration generated by B_t .

2. INSIDER TRADING WITH FULL INFORMATION

The problem of discerning the strategies of a dishonest trader who possesses privileged information in a financial market, “the insider”, is a venerable one in the field of stochastic analysis applied to finance [2, 3, 5, 11, 14, 16] and continues to be of current interest [6, 7, 8]. Within this work, a much simplified version of this problem is considered, as our goal is to favor the accessibility to the comparison between the two anticipating stochastic integrals in the context of finance.

Consider now that, contrary to the situation in the previous section, our trader is an insider with full information on the future price of the stock. Precisely, the insider trader knows already at the initial time $t = 0$ what will the value $S_1(T)$ be. Then the chosen strategy should be different:

$$M_0 = M \mathbb{1}\{\bar{S}_1(T) \leq \bar{S}_0(T)\}, \quad M_1 = M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\},$$

for

$$\begin{aligned} d\bar{S}_0 &= \rho \bar{S}_0 dt, \\ \bar{S}_0(0) &= 1, \end{aligned}$$

and

$$\begin{aligned} d\bar{S}_1 &= \mu \bar{S}_1 dt + \sigma \bar{S}_1 dB_t, \\ \bar{S}_1(0) &= 1, \end{aligned}$$

that is, the insider always bets the most profitable asset. It is then natural to ask what would be the expected wealth of the insider at time T . Note again that we are not considering any utility function modeling risk aversion. This is, as mentioned in the Introduction, in part for the sake of simplicity and in part for modeling reasons: it is not clear what the role of risk aversion should be in the case of an insider with full information on the future value of the stock. In order to answer this question we note that, while the initial value problem

$$\begin{aligned} (6a) \quad dS_0 &= \rho S_0 dt, \\ (6b) \quad S_0(0) &= M \mathbb{1}\{\bar{S}_1(T) \leq \bar{S}_0(T)\}, \end{aligned}$$

is an ordinary differential equation with a random initial condition, the problem

$$\begin{aligned} dS_1 &= \mu S_1 dt + \sigma S_1 dB_t, \\ S_1(0) &= M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\}, \end{aligned}$$

is ill-posed as an Itô stochastic differential equation. This is because the initial condition is anticipating, and this anticipating character will propagate into the solution, therefore giving rise to the Itô integral of a non-adapted integrand, which is of course meaningless. One way to circumvent this pitfall is replacing the Itô integral in our model by one of its generalizations that admit non-adapted integrands. Two possibilities are the Skorokhod integral [18] and the Russo-Vallois forward integral [17]. Both integrals reduce to the Itô one when the integrand is adapted, but are different in general [5].

Using established notation [5], and choosing the Skorokhod integral, we arrive at the initial value problem

$$\begin{aligned} (7a) \quad \delta S_1 &= \mu S_1 dt + \sigma S_1 \delta B_t \\ (7b) \quad S_1(0) &= M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\}, \end{aligned}$$

for a Skorokhod stochastic differential equation. Analogously, when the choice is the Russo-Vallois integral, we face the initial value problem

$$\begin{aligned} (8a) \quad d^- S_1 &= \mu S_1 dt + \sigma S_1 d^- B_t \\ (8b) \quad S_1(0) &= M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\}, \end{aligned}$$

for a forward stochastic differential equation. As happened with the Itô versus Stratonovich dilemma described in the Introduction, it is in principle possible to choose either equation (7a) or (8a) to address the problem at hand. As in this classical situation, both equations (7a) and (8a) are well-founded theoretically [4, 5, 11, 14], so only the particular applications will dictate which is the “right interpretation of noise”. Since we are addressing a financial problem, we will unveil the right choice in this concrete case. To this end we need the following result that describes the time behavior of systems (7a)-(7b) and (8a)-(8b). We remind the reader that the total wealth of the insider is still given by

$$S(t) = S_0(t) + S_1(t).$$

Theorem 2.1. *The expected value of the total wealth of the insider at time $t = T$ is*

$$\begin{aligned}\mathbb{E}[S(T)] &= \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\ &\quad + \frac{M}{2} \left\{ 1 - \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T},\end{aligned}$$

for model (6a)-(6b) and (7a)-(7b), while it is

$$\begin{aligned}\mathbb{E}[S(T)] &= \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\ &\quad + \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 - 2\rho + 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T},\end{aligned}$$

for model (6a)-(6b) and (8a)-(8b), where

$$\operatorname{erf}(\cdot) = \frac{2}{\sqrt{\pi}} \int_0^\cdot e^{-x^2} dx$$

is the error function.

Proof. Using Malliavin calculus techniques [5] it is possible to solve problem (7a)-(7b) explicitly to find

$$S_1(t) = M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\} \diamond \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\},$$

where \diamond denotes the Wick product. Now, using the factorization property of the expectation of a Wick product of random variables, we find for the expected wealth at the terminal time:

$$\begin{aligned}\mathbb{E}[S(T)] &= \mathbb{E}[S_0(T)] + \mathbb{E}[S_1(T)] \\ &= M \mathbb{E}[\mathbb{1}\{\bar{S}_1(T) \leq \bar{S}_0(T)\}] e^{\rho T} \\ &\quad + M \mathbb{E} \left[\mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\} \diamond \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\ &= M \mathbb{E}[\mathbb{1}\{\bar{S}_1(T) \leq \bar{S}_0(T)\}] e^{\rho T} \\ &\quad + M \mathbb{E}[\mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\}] \mathbb{E} \left[\exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\ &= M \Pr \{ \bar{S}_1(T) \leq \bar{S}_0(T) \} e^{\rho T} + M \Pr \{ \bar{S}_1(T) > \bar{S}_0(T) \} e^{\mu T} \\ &= M \Pr \{ B_T \leq (\rho - \mu + \sigma^2/2)T/\sigma \} e^{\rho T} + M \Pr \{ B_T > (\rho - \mu + \sigma^2/2)T/\sigma \} e^{\mu T} \\ &= \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\ &\quad + \frac{M}{2} \left\{ 1 - \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T},\end{aligned}$$

where we have also used that $B_T \sim \mathcal{N}(0, T)$.

Since the forward integral preserves Itô calculus [5], the solution to problem (8a)-(8b) can be computed using Itô calculus rules:

$$S_1(t) = M \mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right\}.$$

Therefore the expected wealth at the terminal time in this case is

$$\begin{aligned}
\mathbb{E}[S(T)] &= \mathbb{E}[S_0(T)] + \mathbb{E}[S_1(T)] \\
&= M \mathbb{E} [\mathbb{1}\{\bar{S}_1(T) \leq \bar{S}_0(T)\}] e^{\rho T} \\
&\quad + M \mathbb{E} \left[\mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&= M \Pr\{\bar{S}_1(T) \leq \bar{S}_0(T)\} e^{\rho T} \\
&\quad + M \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T \right\} \mathbb{E} [\mathbb{1}\{\bar{S}_1(T) > \bar{S}_0(T)\} \exp \{\sigma B_T\}] \\
&= M \Pr\{B_T \leq (\rho - \mu + \sigma^2/2)T/\sigma\} e^{\rho T} \\
&\quad + M \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T \right\} \mathbb{E} [\mathbb{1}\{B_T > (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \{\sigma B_T\}] \\
&= \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\
&\quad + \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 - 2\rho + 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T}.
\end{aligned}$$

□

3. CONSEQUENCES

The problem of insider trading has been approached by means of the use of the forward integral [3, 5, 6, 7, 8, 11, 14], but the justification of this choice has been usually made on more technical grounds. The following result further supports this choice but it is purely based on financial consequences.

Theorem 3.1. *Lets denote by $S^{(i)}(t)$ the total wealth process corresponding to the initial value problems (4) subject to $M_0 = 0$ and (5) subject to $M_1 = M$; denote also by $S^{(sk)}(t)$ and $S^{(rs)}(t)$ the total wealth processes corresponding to the initial value problems (6a)-(6b) and (7a)-(7b), and (6a)-(6b) and (8a)-(8b), respectively. Then*

$$\mathbb{E}[S^{(sk)}(T)] < \mathbb{E}[S^{(i)}(T)] < \mathbb{E}[S^{(rs)}(T)],$$

for any $M, \rho, \mu, \sigma, T \in \mathbb{R}^+$ with $\mu > \rho$.

Proof. From our previous results it is clear that

$$\mathbb{E}[S^{(i)}(T)] = M e^{\mu T}$$

and

$$\begin{aligned}
\mathbb{E}[S^{(sk)}(T)] &= \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\
&\quad + \frac{M}{2} \left\{ 1 - \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T}.
\end{aligned}$$

The inequality

$$\begin{aligned}
M e^{\mu T} &> \frac{M}{2} \left\{ 1 + \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\rho T} \\
&\quad + \frac{M}{2} \left\{ 1 - \operatorname{erf} \left[\frac{(\sigma^2 + 2\rho - 2\mu)\sqrt{T}}{2\sqrt{2}\sigma} \right] \right\} e^{\mu T},
\end{aligned}$$

whenever $\mu > \rho$, follows directly from the definition of the error function [1].

On the other hand, from the proof of Theorem 2.1 we find that

$$\begin{aligned}
\mathbb{E}[S^{(\text{rs})}(T)] &= M \mathbb{E} [\mathbb{1}\{B_T \leq (\rho - \mu + \sigma^2/2)T/\sigma\} e^{\rho T}] \\
&\quad + M \mathbb{E} \left[\mathbb{1}\{B_T > (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&= M \mathbb{E} [\mathbb{1}\{B_T < (\rho - \mu + \sigma^2/2)T/\sigma\} e^{\rho T}] \\
&\quad + M \mathbb{E} \left[\mathbb{1}\{B_T > (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&> M \mathbb{E} \left[\mathbb{1}\{B_T < (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&\quad + M \mathbb{E} \left[\mathbb{1}\{B_T > (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&= M \mathbb{E} \left[\mathbb{1}\{B_T \leq (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&\quad + M \mathbb{E} \left[\mathbb{1}\{B_T > (\rho - \mu + \sigma^2/2)T/\sigma\} \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&= M \mathbb{E} \left[\exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right\} \right] \\
&= M e^{\mu T}.
\end{aligned}$$

□

Our results illustrate that the forward integral provides results with a clear financial meaning, at least in this context. On the other hand the Skorokhod integral yields a result that is meaningless from the financial viewpoint, as the expected wealth of the insider at the terminal time under this model is less than the corresponding wealth of the honest trader.

4. OUTLOOK

Making precise a stochastic differential equation model by means of choosing a suitable stochastic integral is a topic that has received much attention in the physical literature [12]. This choice does not usually change the well-posedness of the problem, but may modify abruptly the dynamics of the equation. Therefore the selection should be based on modeling assumptions, and of course any particular choice is strongly model-dependent. While historically the discussion has focused on the non-anticipating framework and the Itô/Stratonovich duality, there is nothing substantially different between this case and the anticipating one in this respect. Therefore the question of interpreting a given anticipating stochastic differential equation in the Skorokhod or Russo-Vallois sense falls in this category. Our present results point to the fact that the Russo-Vallois forward integral is well-adapted for modeling insider trading in a financial market, but the Skorokhod integral is not suitable for this purpose. This of course does not affect the fact that both types of anticipating stochastic differential equation are well-defined, and that presumably the “Skorokhod interpretation of noise” will be of use in other applications, be them financial, physical, or yet others. Time will reveal which anticipating stochastic integrals are useful in different applications, just like the applications the Itô and Stratonovich stochastic integrals are useful for have been revealed along the years.

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